Функции нескольких переменных
Частные производные

Задание.

Для функции $z(x,y)$ *двух переменных, неявно заданной уравнением* $\sin(xz)+\cos(yz)=1$, *найдите первый и второй дифференциалы в точке* $x=y=1, z=0$.

Решение. Введем функцию $F(x,y,z)=\sin(xz)+\cos(yz)-1=0$. Найдем ее частные производные:

$$F_x^{'}=\left(\sin(xz)+\cos(yz)-1\right)_x^{'}=z\cos(xz),$$

$$F_y^{'}=\left(\sin(xz)+\cos(yz)-1\right)_y^{'}=-z\sin(yz),$$

$$F_z^{'}=\left(\sin(xz)+\cos(yz)-1\right)_z^{'}=x\cos(xz)-y\sin(yz).$$

Тогда

$$\frac{\partial z}{\partial x}=-\frac{F_x^{'}}{F_z^{'}}=-\frac{z\cos(xz)}{x\cos(xz)-y\sin(yz)},$$

$$\frac{\partial z}{\partial y}=-\frac{F_y^{'}}{F_z^{'}}=-\frac{-z\sin(yz)}{x\cos(xz)-y\sin(yz)}=\frac{z\sin(yz)}{x\cos(xz)-y\sin(yz)}$$

В точке $x=y=1, z=0$:

$\frac{\partial z}{\partial x}(1;1;0)=0$, $\frac{\partial z}{\partial y}(1;1;0)=0$, дифференциал равен: $dz(1;1;0)=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=0$.

Найдем второй дифференциал. Вычисляем вторые производные.

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\left(\frac{z\cos(xz)}{x\cos(xz)-y\sin(yz)}\right)'_x=$$

$$=\frac{z\cos(xz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(x\cos(xz)-y\sin(yz)\right)'_x-\frac{\left(z\cos(xz)\right)'_x}{x\cos(xz)-y\sin(yz)}=$$

$$=\frac{z\cos(xz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(\cos(xz)-x\sin(xz)\left(z+xz'_x\right)-y\cos(yz)\,yz'_x\right)$$

$$-\frac{\left(z'_x\cos(xz)-z\sin(xz)\left(z+xz'_x\right)\right)}{x\cos(xz)-y\sin(yz)}.$$

В точке $x=y=1, z=0$ получаем: $\frac{\partial^2 z}{\partial x^2}=0-0=0.$

Далее:

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-\left(\frac{z\cos(xz)}{x\cos(xz)-y\sin(yz)}\right)'_y=$$

$$=\frac{z\cos(xz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(x\cos(xz)-y\sin(yz)\right)'_y-\frac{\left(z\cos(xz)\right)'_y}{x\cos(xz)-y\sin(yz)}=$$

$$=\frac{z\cos(xz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(-x\sin(xz)\,xz'_y-\sin(yz)-y\cos(yz)\left(z+yz'_y\right)\right)$$

$$-\frac{\left(z'_y\cos(xz)-z\sin(xz)\,xz'_y\right)}{x\cos(xz)-y\sin(yz)}.$$

В точке $x=y=1, z=0$ получаем: $\frac{\partial^2 z}{\partial x\partial y}=0-0=0.$

Найдем последнюю производную:

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{z\sin(yz)}{x\cos(xz)-y\sin(yz)}\right)_y^{'}=$$

$$=-\frac{z\sin(yz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(x\cos(xz)-y\sin(yz)\right)_y^{'}+\frac{\left(z\sin(yz)\right)_y^{'}}{x\cos(xz)-y\sin(yz)}=$$

$$=-\frac{z\sin(yz)}{\left(x\cos(xz)-y\sin(yz)\right)^2}\left(-x\sin(xz)xz_y^{'}-\sin(yz)-y\cos(yz)\left(z+yz_y^{'}\right)\right)+$$

$$+\frac{\left(z_y^{'}\sin(yz)+z\cos(yz)\left(z+yz_y^{'}\right)\right)}{x\cos(xz)-y\sin(yz)}.$$

В точке $x=y=1, z=0$ получаем: $\frac{\partial^2 z}{\partial y^2}=0-0=0.$

Таким образом, второй дифференциал также равен нулю в данной точке:

$(dz)^2(1;1;0)=0$