# Операционное исчисление
# Решение интегрального уравнения

Задание.

Методом операционного исчисления найти решение интегрального уравнения

$$y(t) = \cos t + \int_0^t (t-\tau)^2 y(\tau)d\tau$$

Решение.

Воспользуемся теоремой об умножении изображений:

$$\int_0^t f(\tau)g(t-\tau)d\tau \doteqdot F(p)G(p)$$

Пусть $y(t)$ - оригинал, $y(t) \doteqdot Y(p)$, тогда

$$t^2 \doteqdot \frac{2}{p^3}$$

$$\int_0^t (t-\tau)^2 y(\tau)d\tau \doteqdot \frac{2}{p^3} \cdot Y(p)$$

Так как

$$\cos t \doteqdot \frac{p}{p^2+1},$$

уравнение примет вид

$$Y(p) = \frac{p}{p^2+1} + \frac{2}{p^3} \cdot Y(p)$$

Решим уравнение относительно $Y(p)$:

$$Y(p) \cdot \left(1 - \frac{2}{p^3}\right) = \frac{p}{p^2+1}; \quad Y(p) \cdot \left(\frac{p^3-2}{p^3}\right) = \frac{p}{p^2+1}$$

$$Y(p) = \frac{p^4}{(p^2+1)(p^3-2)}$$

Теперь по полученному изображение восстановим оригинал.

$$\frac{p^4}{(p^2+1)(p^3-2)} = \frac{Ap+B}{p^2+1} + \frac{Cp^2+Dp+E}{p^3-2} =$$

$$= \frac{(A+C)p^4 + (B+D)p^3 + (C+E)p^2 + (-2A+D)p + (-2B+E)}{(p^2+1)(p^3-2)}$$

$$\begin{cases} A+C=1 \\ B+D=0 \\ C+E=0 \\ -2A+D=0 \\ -2B+E=0 \end{cases};\quad A=\frac{1}{5};B=-\frac{2}{5};C=\frac{4}{5};D=\frac{2}{5};E=-\frac{4}{5}$$

$$\frac{p^4}{(p^2+1)(p^3-2)}=\frac{1}{5}\cdot\frac{p-2}{p^2+1}+\frac{2}{5}\cdot\frac{2p^2+p-2}{p^3-2}$$

$$\frac{2p^2+p-2}{p^3-2}=\frac{2p^2+p-2}{\left(p-2^{\frac{1}{3}}\right)\left(p^2+2^{\frac{1}{3}}p+2^{\frac{2}{3}}\right)}=\frac{A}{p-2^{\frac{1}{3}}}+\frac{Bp+C}{p^2+2^{\frac{1}{3}}p+2^{\frac{2}{3}}}=$$

$$=\frac{(A+B)p^2+\left(2^{\frac{1}{3}}A-2^{\frac{1}{3}}B+C\right)p+\left(2^{\frac{2}{3}}A-2^{\frac{1}{3}}C\right)}{\left(p-2^{\frac{1}{3}}\right)\left(p^2+2^{\frac{1}{3}}p+2^{\frac{2}{3}}\right)}$$

$$\begin{cases} A+B=2 \\ 2^{\frac{1}{3}}A-2^{\frac{1}{3}}B+C=1 \\ 2^{\frac{2}{3}}A-2^{\frac{1}{3}}C=-2 \end{cases}\rightarrow\quad \begin{aligned} A&=\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}} \\ B&=\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}} \\ C&=\frac{1}{3}+\frac{2}{3}2^{\frac{1}{3}}+\frac{2}{3}2^{\frac{2}{3}} \end{aligned}$$

$$\frac{2p^2+p-2}{p^3-2}=\frac{\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)}{p-2^{\frac{1}{3}}}+\frac{\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)p+\left(\frac{1}{3}+\frac{2}{3}2^{\frac{1}{3}}+\frac{2}{3}2^{\frac{2}{3}}\right)}{p^2+2^{\frac{1}{3}}p+2^{\frac{2}{3}}}$$

$$\frac{\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)p+\left(\frac{1}{3}+\frac{2}{3}2^{\frac{1}{3}}+\frac{2}{6}2^{\frac{2}{3}}\right)}{p^2+2^{\frac{1}{3}}p+2^{\frac{2}{3}}}$$

$$=\frac{\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)p+\left(\frac{1}{3}+\frac{2}{3}2^{\frac{1}{3}}+\frac{2}{3}2^{\frac{2}{3}}\right)}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}=$$

$$=\frac{\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)\left(p+2^{-\frac{2}{3}}\right)+\left(\frac{1}{2}+\frac{1}{2}2^{\frac{2}{3}}\right)}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}$$

$$=\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)\cdot\frac{\left(p+2^{-\frac{2}{3}}\right)}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}+$$

$$+\frac{1}{3}\left(2+2^{\frac{1}{3}}\right)\cdot\frac{\frac{3}{4}2^{\frac{2}{3}}}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}$$

Окончательно имеем

$$Y(p)=\frac{p^4}{(p^2+1)(p^3-2)}=\frac{1}{5}\cdot\frac{p-2}{p^2+1}+\frac{2}{5}\cdot\frac{2p^2+p-2}{p^3-2}=$$

$$=\frac{1}{5}\cdot\frac{p-2}{p^2+1}+\frac{2}{5}\cdot\frac{\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)}{p-2^{\frac{1}{3}}}+\frac{2}{5}\cdot\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)\cdot\frac{\left(p+2^{-\frac{2}{3}}\right)}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}$$

$$+$$

$$+\frac{2}{15}\cdot\left(2+2^{\frac{1}{3}}\right)\cdot\frac{\frac{3}{4}2^{\frac{2}{3}}}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}$$

Получим оригинал:

$$\frac{1}{5}\cdot\frac{p-2}{p^2+1}=\frac{1}{5}\cdot\frac{p}{p^2+1}-\frac{2}{5}\cdot\frac{1}{p^2-1}\doteqdot\frac{1}{5}\cos t-\frac{2}{5}\sin t$$

$$\frac{2}{5}\cdot\frac{\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)}{p-2^{\frac{1}{3}}}\doteqdot\frac{2}{5}\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)e^{2^{\frac{1}{3}}}$$

$$\frac{2}{5}\cdot\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)\cdot\frac{\left(p+2^{-\frac{2}{3}}\right)}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}$$

$$\doteqdot\frac{2}{5}\cdot\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)e^{-2^{-\frac{2}{3}}}\cos\left(\frac{\sqrt{3}}{2}\cdot 2^{1/3}\right)$$

$$\frac{2}{15}\cdot\left(2+2^{\frac{1}{3}}\right)\cdot\frac{\frac{3}{4}2^{\frac{2}{3}}}{\left(p+2^{-\frac{2}{3}}\right)^2+\frac{3}{4}2^{\frac{2}{3}}}\doteqdot\frac{2}{15}\cdot\left(2+2^{\frac{1}{3}}\right)e^{-2^{-\frac{2}{3}}}\sin\left(\frac{\sqrt{3}}{2}\cdot 2^{1/3}\right)$$

Итак,

$$y(t)=\frac{1}{5}\cos t-\frac{2}{5}\sin t+\frac{2}{5}\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)e^{2^{\frac{1}{3}}}+\frac{2}{5}$$
$$\cdot\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)e^{-2^{-\frac{2}{3}}}\cos\left(\frac{\sqrt{3}}{2}\cdot 2^{\frac{1}{3}}\right)+$$

$$+\frac{2}{15}\cdot\left(2+2^{\frac{1}{3}}\right)e^{-2^{-\frac{2}{3}}}\sin\left(\frac{\sqrt{3}}{2}\cdot 2^{1/3}\right)$$

ОТВЕТ.

$$y(t)=\frac{1}{5}\cos t-\frac{2}{5}\sin t+\frac{2}{5}\left(\frac{2}{3}-\frac{1}{3}2^{\frac{1}{3}}+\frac{1}{6}2^{\frac{2}{3}}\right)e^{2^{\frac{1}{3}}}+\frac{2}{5}$$
$$\cdot\left(\frac{4}{3}+\frac{1}{3}2^{\frac{1}{3}}-\frac{1}{6}2^{\frac{2}{3}}\right)e^{-2^{-\frac{2}{3}}}\cos\left(\frac{\sqrt{3}}{2}\cdot 2^{\frac{1}{3}}\right)+$$
$$+\frac{2}{15}\cdot\left(2+2^{\frac{1}{3}}\right)e^{-2^{-\frac{2}{3}}}\sin\left(\frac{\sqrt{3}}{2}\cdot 2^{1/3}\right)$$